Semantic Assumptions in the Philosophy of Mathematics*

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Abstract

The standard semantic analysis of sentences such as 'The number of planets in the solar system is eight' is that they are identity statements that identify certain mathematical objects, namely numbers. The analysis thereby facilitates arguments for a controversial philosophical position, namely realism about mathematical objects. Whether or not this analysis is accurate should concern philosophers greatly. Recently, several authors have offered rival analyses of sentences such as these. I consider a wide range of linguistic evidence to show that all of these analyses, including the standard analysis, suffer significant drawbacks. I then outline and present further evidence in favour of my own analysis, developed elsewhere, according to which such sentences are identity statements that identify certain kinds of facts. I also defend a novel and plausible approach to the semantics of interrogative clauses that corroborates my analysis. Finally, I discuss how realists about mathematical objects should proceed in light of the arguments presented in this paper.

1. Introduction

The following sentences are true:

(1a) The number of planets in the solar system is eight.

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(1b) The mass of Jupiter in kilograms is 1.8986×10^{27}.

This observation appears harmless enough, but such observations facilitate important and controversial arguments in the philosophy of mathematics. For example, if we adopt the standard assumption that the mathematical expressions in (1a-b) stand for mathematical objects, then in order for (1a-b) to be true, those mathematical objects must exist. From this, we get a so-called 'easy argument' for realism about mathematical objects.

Further, sentences such as (1a-b) are indispensable to our best scientific theories. In spite of Hartry Field's (1980, 1989) heroic attempts to show otherwise, we cannot do science without employing arithmetical language. Add to this the premise that we should believe in all those entities apparent reference to which is indispensable to our best scientific theories, and we have the foundations of the indispensability argument for realism about mathematical objects.

Both of these arguments rely on the substantial semantic thesis that sentences such as (1a-b) contain expressions that refer to numbers. The standard analysis of (1a-b) assumed in much of the philosophy of mathematics says that number-of expressions and magnitude-of expressions, such as 'The number of planets in the solar system' and 'The mass of Jupiter in kilograms', as well as numerals, such as 'eight' and '1.8986×10^{27}', refer to numbers. On this view, sentences such as (1a-b) are naturally interpreted as expressing the identity of the referents of the pre- and post-copular expressions. Because it is inspired by the work of Frege ([1884] 1953: 69), I will call this analysis the *Fregean analysis*. I should distinguish here between the Fregean analysis itself and the spirit in which it can be put forward. This is important, since Frege arguably did not develop the analysis in order to account for the semantics of natural language, but instead as part of his project of developing an ideal language for mathematics and science. I am concerned with the Fregean analysis as put forward as a thesis about natural language.

It is easy to see how the Fregean analysis facilitates easy arguments. If (1a) states that the number of planets is identical with the number eight, then its truth-conditions demand that the number eight exists. Hence, from the plausible premise that (1a) is true, we reach the conclusion that there are numbers.

The way in which indispensability arguments rely on the Fregean analysis requires further explanation. The most influential, and arguably the first, presentation of the indispensability argument is Quine (1948). For Quine, to establish what a theory says there is we must first translate the theory into a canonical language. For familiar reasons, first-order predicate logic is chosen for this task. Then we establish which objects the bound variables of the theory must range over in order for the theory to be true. Those objects are what the theory is committed to. Indispensability arguments rest on the principle that the entities that our best scientific theories are committed to in this way exist.

For the indispensability argument presented above to go through, it must be assumed that the indispensability of sentences such as (1a-b) to our best scientific theories implies that numbers must be among the values of the bound variables in our canonical formulations of these sentences in order for the theories they comprise to be true. This assumption is borne out by the procedure used to translate our scientific theories into the canonical language, and, in particular, how we translate referring expressions. Quine does not propose that we introduce a constant into the language for each referring expression, but instead shows how to capture the truth-conditions of sentences containing referring expressions with just predicates, variables and first-order quantifiers. To take the famous example, 'Pegasus flies' is not translated into 'Fp', where 'p' is a constant that refers to Pegasus and 'F' is the predicate '... flies'. Instead, it is translated as '$\exists x(Px \wedge \forall y(Py \rightarrow x = y) \wedge Fx)$', where '$P$' is the predicate '... pegasizes'. For a theory containing this sentence to be true, Pegasus must be among the values for the bound variables, and so a theory containing this sentence is committed to the existence of Pegasus.

The above indispensability argument therefore assumes that sentences such as (1a-b) contain expressions that refer to numbers. Without this assumption, there is no reason to think that, for the first-order counterparts of (1a-b) to be true, a number must serve as the value of the bound variables.

We have seen that the Fregean analysis facilitates arguments for a controversial philosophical position. Accordingly, whether or not this analysis is accurate should concern philosophers greatly. Recently, several authors (Hofweber 2005; Moltmann 2013a, 2013b, 2016; and Felka 2014) have offered rival analyses of sentences such as (1a-b). In what follows, I will consider a wide range of linguistic evidence demonstrating that all of these analyses suffer significant drawbacks. I will then outline and present evidence in favour of an alternative analysis.

In §2, I present a syntactico-semantic puzzle, *Frege's Other Puzzle*, that any plausible analysis of (1a-b) must provide a solution to, and show that the Fregean analysis fails to do so. I endorse an existing attempt to solve the semantic dimension of this puzzle, which involves arguing that the post-copular expressions of (1a-b) are not referring expressions. I show that evidence provided by Friederike Moltmann in favour of this claim is inconclusive, but provide my own evidence that is more suggestive. In §3, I discuss evidence presented by Moltmann that is supposed to show two things: first, that the pre-copular expressions in (1a-b) do not purport to refer to numbers; second, that they do purport to refer to tropes. I show that her evidence is only suggestive of the former conclusion and provide some evidence to suggest that number-of and magnitude-of expressions instead refer to facts. In §4, I present a recent syntactic analysis of sentences such as (1a-b) as specificational sentences, comprised of question and answer pairs, that promises to solve the syntactic dimension of Frege's Other Puzzle. Though I agree that (1a-b) are specificational, I raise some issues with the question-answer analysis. In §5, I argue for a particular analysis of questions and answers that corroborates two claims of the preceding sections: that the received semantic analysis of specificational sentences is wrong; and that the pre-copular expressions in (1a-b) refer to facts. In §6, I present a satisfactory semantic analysis of specificational sentences, and so of (1a-b). I show that philosophers cannot appeal to the truth-conditions of sentences such as (1a-b) alone in order to establish realism about mathematical objects. I end by suggesting ways in which the realists might proceed in light of this: they must either supplement their arguments with metaphysical arguments, or appeal to different areas of mathematical language. I should note that, in what follows, I will use '?' to indicate that a sentence is ungrammatical, and '*' to indicate that a sentence is semantically deviant, or otherwise infelicitous.

2. Frege's Other Puzzle and the Semantics of Numerals

Thomas Hofweber (2005: 179-180) identifies a puzzle for the Fregean analysis. Numerals can occupy two syntactic positions in natural language sentences. In (1a), the cardinal 'eight' is in singular-term position, paradigmatically occupied by names, such as 'Dostoyevsky':

(2) The author of *Crime and Punishment* is Dostoyevsky.

Contrast (3a-b), in which 'eight' is in adjectival position, typically reserved for adjectives, such as 'red':

(3a) There are eight planets in the solar system.

(3b) Jupiter has a mass of 1.8986×10^{27} kilograms.

(3c) There are red planets in the solar system.

The puzzle has a syntactic and semantic dimension. The syntactic dimension is that grammar forbids referring expressions from occurring in adjectival position. For example, replacing 'eight', '1.8986×10^{27}', and 'red' in (3a-c) with 'The number of planets in the solar system', 'The mass of Jupiter in kilograms', and 'The colour of blood', respectively, yields ungrammatical results:

(4a) ? There are the number of planets in the solar system planets. (Adapted from Hofweber 2005: 179)

(4b) ? Jupiter has a mass of the mass of Jupiter in kilograms kilograms.

(4c) ? There are the colour of blood planets in the solar system.

Intuitively, 'The number of planets in the solar system', 'The mass Jupiter in kilograms' and 'The colour of blood' cannot take adjectival position because they are referring expressions, making it all the more puzzling that 'eight' can.

The semantic dimension of the puzzle arises because numerals in adjectival position also appear to have a different semantic function to their counterparts in singular-term position: they do not appear to stand for an object. An account of the relation between the semantic function of both occurrences must be provided. Following Hofweber (2005: 180), I call this syntactico-semantic puzzle 'Frege's Other Puzzle'

(FOP). (Again, given Frege's project, I do not wish to imply that Frege himself was or should have been concerned with this puzzle; this is a puzzle for the Fregean analysis put forward as a thesis about natural language semantics.)

Hofweber (2005), Moltmann (2013a; 2013b; 2016) and Katharina Felka (2014) have motivated analyses of sentences such as (1a-b) by claiming that they solve FOP. A semantic theory that solves FOP is to be preferred to the Fregean analysis only if it provides the best explanation of all the other available linguistic data. We shall see that none of these analyses do both.

Though I will reject their analyses, I endorse the spirit of these authors' solutions to the semantic dimension of FOP, which involves arguing that the post-copular expressions in (1a-b) are not referring expressions.

Generalised quantifier theory (GQT) yields a plausible interpretation of numerals in adjectival position as determiners. (GQT is based on the work of Mostowski 1957 and Montague 1974, and has subsequently been developed by many theorists. See Keenan and Westerstahl 1997 for an overview.) Along with a demonstration that numerals function as determiners in contexts such as (1a-b), this is sufficient to solve the semantic dimension of FOP. (However, see Knowles 2015a: 2758-2762 for a GQT-based account of numerals as adjectives. The overall argument of this paper does not rest on which specific semantic function is assigned to the numerals in (1a-b), but only that they do not serve as referring expressions.)

In the remainder of this section, I will do three things. First, I will now outline the GQT account of numerals. Second, I will show that evidence presented by Moltmann in support of the claim that the numerals occurring in (1a-b) are not referring expressions is inconclusive. Third, I will present my own evidence for this claim that is more suggestive.

Analysing natural language using the quantifiers of predicate calculus is problematic. 'Something' is assigned the existential quantifier and 'Everything' the universal; these are the basic units of analysis. But in English there are more complex quantifier phrases, such as the following: 'Some apples'; 'Every man'; 'Few coins' etc. This suggests that 'Something' and 'Everything' are also complex, made up of 'Some', 'Every' and 'Thing'. (I am grateful to an anonymous reviewer for pointing out that the universal quantifier in other languages, such as French and German, does not behave in this way). There are also quantifier phrases in natural language that resist analysis in terms of the existential and universal quantifiers ('Most', for example). An adequate analysis of natural language should provide a unified and systematic analysis of quantifier phrases (Hofweber 2005: 196), and account for all the quantifier phrases of the language (Barwise and Cooper 1981: 159-61).

GQT provides such an account. According to GQT, sentences are typically composed of two syntactic types: a verb phrase (VP) and a noun phrase (NP). Sets are the semantic values of VPs and functions from sets to truth-values are the semantic values of NPs. The set of things that run fast is assigned to 'run fast'; assigned to 'Adult cheetahs' is a function that yields true for all and only sets that contain the adult cheetahs. Thus, 'Adult cheetahs run fast' is true iff the adult cheetahs are members of the set of things that run fast; that is, true iff adult cheetahs run fast.

NPs can be complex, consisting of a noun and a modifier. Determiners modify nouns or NPs to help determine the extent of their reference. 'Some' in 'Some apples' modifies 'apples' and determines reference to at least one apple. Other determiners include 'Most', 'Every' and 'Both'. Quantified NPs are complex NPs whose modifying element is a determiner. The semantic value of a noun is a set so the semantic value of a determiner is a function from sets to functions from sets to truth-values.

According to GQT, numerals in adjectival position are determiners. They take NPs as arguments and determine their reference: 'Six' determines reference to six apples in 'Six apples'. This is a plausible and widely accepted semantics for determiners that yields a plausible semantic account of numerals in adjectival position. Recall (1a-b) and (3a-b):

(1a) The number of planets in the solar system is eight.

(1b) The mass of Jupiter in kilograms is 1.8986×10^{27}.

(3a) There are eight planets in the solar system.

(3b) Jupiter has a mass of 1.8986×10^{27} kilograms.

Either 'eight' and '1.8986×10^{27}' must be shown to have related semantic functions across these two contexts, or the difference must be plausibly and systematically explained. After all, the numerals in (1a-b) and the numerals in (3a-b) are not plausibly homonyms. To meet this challenge, Moltmann (2013a: 522; 2016) provides evidence for the claim that sentences such as (1a-b) are not identity statements, thus implying that their post-copular expressions are not a referring expressions:

(5) * The number of planets in the solar system is the number eight.

If 'The number eight' and 'eight' are both singular terms standing for the number eight, they should be substitutable salva congruitate and salva veritate. However, (5) sounds odd, and the truth of (1a) is not obviously preserved in it. The supposed explanation of these intuitions is that the two expressions belong to different semantic types. If 'eight' in (5) is a determiner, for example, then the unacceptability of the substitution would be expected.

This evidence is inconclusive for two reasons. First, there is a simpler explanation: (5) is a pleonastic sentence. The repetition of information in the latter part could suggest it is just a long-winded version of (1a). Compare:

(6a) The metal of the bike is steel.

(6b) The metal of the bike is the metal steel.

(6b) is odd because it repeats information, and is rarely, if ever, used. It does not follow that 'steel' is of a different semantic type to 'the metal steel'; quite the opposite: the former is the latter abbreviated. This can explain why (5) sounds odd to us, and its oddness, in turn, can explain why it is not obvious that the truth of (1a) is preserved.

Second, the principle that Moltmann's evidence relies on, that co-referring expressions must be directly substitutable salva congruitate (call it the *naïve reference principle*) has been discredited (see Oliver 2005, among others). Co-referring expressions are often not substitutable, so (5) does no more than further demonstrate the inadequacy of the naïve reference principle.

However, David Dolby (2009) convincingly defends a modified version of the principle: co-referring expressions are substitutable by a process of generalisation and specification in accordance with the rules of grammar. Call this the *reference principle*. I will now defend this principle, and use it to provide more compelling evidence that suggests numerals do not behave as referring expressions in contexts such as (1a-b), and so lends support to the analysis of them as determiners.

Dolby helpfully categorises failures of direct substitution into three categories: first, those involving complex referring expressions; second, those problematic because of the inflection of language; and third, those involving pre-modifying adjectives. I will concentrate on the third, most relevant kind. To use Dolby's example, 'Multi-cultural' occurs as a pre-modifying adjective in (7a):

(7a) Multi-cultural Britain has draconian restrictions on free speech.

(7b) ? Multi-cultural the land of Churchill has draconian restrictions on free speech.

Replacing the co-referring 'The land of Churchill' for 'Britain' gives the ungrammatical (7b).

Dolby claims that co-referring expressions often cannot be directly substituted because the rules governing substitution are more complicated than the naïve reference principle implies: 'the rules according to which competent speakers make substitutions are also the rules for the formation of generalizations from particular statements and for the specification of these generalizations' (2009: 90).

Dolby's rules of generalisation and specification are as follows. The singular term is replaced with a general term (a determiner) in accordance with the rules of grammar:

(8a) Hungry James came home and ate some pasta.

(8b) Someone hungry came home and ate some pasta.

Then the generalisation is specified using a co-referring expression for 'James':

(8c) The hungry student of the house came home and ate some pasta.

This is grammatical, but Dolby anticipates an objection: 'If substitution is to be understood as proceeding according to rules that form grammatical sentences from grammatical sentences, how can any substitution ever fail…?' (2009: 294). Substitution can fail when there are no rules that take us from the sentence to a generalisation and back to a specification including the expression substituted. For instance, there are no rules for substituting 'Silently' for 'the leader of the free world' in any sentence (Dolby 2009: 294-5).

(5) alone does not support the claim that numerals are not referring expressions in contexts such as (1a-b): there are numerous examples of obviously co-referring expressions not substituting directly. However, we have seen that the naïve reference principle does not reflect the rules for substitution in natural language. Substitution failure occurs when there are no rules taking us from an appropriate generalisation to an appropriate specification. For the substitution of 'The number eight' for 'eight' in (1a) to fail, there must be no rules taking us from a generalisation of (1a) to a specification that includes 'The number eight'. I will now argue that there are no such rules.

There are two ways in which we might generalise from (1a-b). First:

(1a′) The number of planets in the solar system is something.

(1b′) The mass of Jupiter in kilograms is something.

To my ear, both sound strange, but even if they are not, we should not be hasty in drawing any conclusions from this. Though 'something' is often apt for being replaced by a referring expression, in some contexts it is not. For example, if Mary and John are virtuous, we can infer:

(9a) There is something that both Mary and John are.

Specifying what Mary and John are by extending the sentence cannot be done using the referring expression 'Virtue':

(9) ? There is something that both Mary and John are, namely Virtue.

With the predicate 'virtuous', however, it can:

(9c) There is something that both Mary and John are, namely virtuous.

This reveals that 'something' is not standing in for a referring expression in (9a) (for more on this 'non-nominal quantification', see Prior 1971: 34-37; Sellars 1960; Künne 2006: 272-9; and Rosefeldt 2008). So, we cannot infer anything from the fact that (1a′) and (1b′) are appropriate generalisations of (1a-b) (if they are) because it is unclear what position 'something' occupies. In contrast, the following equally acceptable generalisations are suggestive that it does not stand in for a singular term:

(1a″) The number of planets in the solar system is so many/however many/ that many.

(1b″) The mass of the rock in kilograms is so many/however many/that many.

Some of these sentences may not strike the reader as particularly idiomatic, but in this context, the fact that they are grammatical and not semantically deviant is enough to be suggestive. 'So many', 'however many' and 'that many' are not plausibly expressions that take the place of referring expressions. Notice that they can all take the adjectival position:

(3a′) There are so many/however many/that many planets in the solar system.

(3b′) Jupiter has a mass of so many/however many/that many kilograms.

There are no rules taking us from (1a″-b″) and (3a′-b′) to sentences containing 'The number twelve' or 'The number 1.8986×10^{27}'. This provides a good reason for thinking that 'eight' and '1.8986×10^{27}' in (1a-b) are not behaving semantically as referring expressions, and, since 'so many' and 'however many' (and perhaps even 'that many') can plausibly be seen as standing in for determiners, this provides some reason for extending the GQT account to contexts such as (1a-b), and so analysing the numerals therein as determiners.

I have provided evidence in favour of a particular solution to the semantic dimension of FOP. The evidence at least suggests that the post-copular expressions in contexts such as (1a-b) are not referring expressions, and goes some way to justifying the claim that they occur as determiners. It appears that the Fregean analysis, put forward as a thesis about the semantics of natural language sentences, is mistaken. However, the syntactic dimension of FOP remains: why can non-referring expressions occur in singular-term position as well as their natural adjectival position? Before moving onto this question, I will now provide good reasons for thinking that the pre-copular expressions of sentences such as (1a-b) do not purport to refer to numbers, though they do purport to refer.

3. Number-of and Magnitude-of Expressions

On the Fregean analysis, number-of expressions and magnitude-of expressions stand for numbers. In this section, I examine evidence presented by Moltmann for the claim that number-of expressions stand instead for number tropes. Tropes are typically understood as particular instances of properties, including the beauty of a particular painting, or the fragility of a particular vase, and are the qualitative aspects of objects. Moltmann claims that number tropes are the quantitative aspects of pluralities.

Moltmann presents two kinds of evidence. The first is meant to show that number-of expressions are referring expressions that do not stand for numbers. To see that 'The number of planets in the solar system' is a singular term, consider:

(10a) The number of planets in the solar system is small.

(10b) The number of planets in the solar system is surprising.

(10c) The number of planets in the solar system is the same as the number of chickens.

We have little choice but to interpret 'The number of planets in the solar system' as standing for an object that each predicate is describing.

Moltmann argues that number-of expressions do not stand for numbers by demonstrating that constructions permissible with number-referring expressions are not permissible with number-of expressions, and vice/versa (2013a: 502-4; 2016). Consider:

(11a) * The number eight is small.

(11b) * The number eight is surprising.

(11c) * The number eight is the same as the number eight.

(11d) The number eight is the number eight.

Though (11c) may be permissible, it is less natural than (11d). (11a) and (11b), though grammatical, make little sense. Similarly, when we rewrite (11c) as a straightforward identity statement, we get:

(12a) * The number of planets is the number of chickens.

This sounds odd, if it makes sense. If the number of pigs and the number of chickens were numbers, we should expect such an identity statement to sound fine.

This leads Moltmann to conclude that number-of expressions do not stand for numbers. However, the evidence is not conclusive. The permissibility of (10a-c) shows that, in those contexts, 'The number of planets in the solar system' doesn't stand for a number: it is that there are eight planets that is surprising. It does not show that number-of expressions do not stand for numbers in other contexts.

Suppose that John typically wants things that are large and non-shiny, but now he wants a diamond. Compare:

(13a) What John wants is unusual.

(13b) What John wants is small and shiny.

In (13a), it is that John's wants the diamond that is said to be unusual, the diamond itself need not be unusual for the sentence to be true. However, (13b) appears to describe the diamond. The same expression, 'What John wants', stands for a different object depending on the predicate used. Similarly, 'The number of planets in the solar system' does not stand for a number in (10a-c), but in certain mathematical contexts, it clearly does:

(14a) The number of planets in the solar system is even.

(14b) The number of planets in the solar system is odd.

Moltmann points out that certain mathematical predicates make unacceptable constructions:

(15b) * The number of planets in the solar system is rational.

(15c) * The number of planets in the solar system is real.

Moreover, she claims there is a unifying feature of mathematical contexts that are permissible: they can be verified or falsified by performing operations on collections of objects (2016). (14a) implies the planets can be divided into two groups of equal number; (14b) implies they can't. Both can be verified or falsified by arranging the objects and counting them. Moltmann claims this is explained by the fact that number-of expressions refer to aspects of collections.

However, there is an alternative explanation for the oddness of (15a-b). The adjectives 'real' and 'rational' are ambiguous between their mathematical and everyday meaning. The former is appropriate for describing numbers, while the latter is appropriate for describing worldly objects. If number-of expressions refer to different things depending on their adjoining predicate, then the strangeness of (15a-c) would be unsurprising: the referent of 'The number of planets in the solar system' depends on the interpretation of the predicate, and the interpretation of the predicate depends on the referent of the referring expression. To dispel the ambiguity, it must be made clear that the predicate is mathematical:

(16a) The number of planets in the solar system is a rational number.

(16b) The number of planets in the solar system is a real number.

This sounds fine because the predicate is unmistakably mathematical, forcing an interpretation of 'The number of planets in the solar system' as standing for a number. Contra Moltmann, it appears that number-of expressions do purport to stand for numbers in some mathematical contexts. Nevertheless, it is clear that in some non-mathematical contexts they stand for something else.

Moltmann's second kind of evidence is supposed to show that the referents of number-of expressions share properties with the relevant collections. Consider:

(17a) The number of women is unusual.

(17b) The women are unusual, in number.

The predicate 'is unusual' can be applied to both the number of women, and the collection of women, so long as the modifier 'in number' is added. Moltmann claims this shows that the collection of women and the number of women share the property of being unusual. She claims that, because 'in number' must be added to preserve the meaning of (17a), it is something to do with the quantitative aspect of the women that is unusual.

This evidence is dubious. It is true that the number of women bears a special relation to the collection of women that allows the move from (17a) to (17b), but this does not mean that these two entities share the property of being unusual. Compare:

(18a) The font of the book is large.

(18b) The book is large, in terms of its font.

The book and the font stand in a relation that permits the move from (18a) to (18b), but the book and the font do not share the property of being large. (18a) and (18b) can be true when the book is small. Similarly, it is the number of women that is unusual; the women need not be.

Suppose these two entities do share properties. It does not immediately follow that number tropes are the ideal candidates for the referents of number-of expressions. Moltmann argues that the kinds of properties shared by the referents of number-of expressions and collections can only be instantiated by concrete entities, and so concludes that the referents of these terms are number tropes, the concrete, quantitative aspects of collections (2016).

The standard notion of concreteness is assumed: an entity is concrete iff it can act as an object of perception, a relatum of causal relations, and is spatio-temporally located. Moltmann points out that perceptual and causal predicates apply to the referents of number-of expressions:

(19a) John noticed the number of women.

(19b) The number of women caused Mary consternation.

This is poor evidence. Examples attributing causal and perceptual properties to things that cannot enter into causal or perceptual relations abound:

(20a) John began to see Mary's point of view.

(20b) Fermat's Last Theorem caused Mary frustration.

Objection: 'see' is figurative in (20a). But that is precisely the point. Perceptual predicates are often used in this way to express that something is understood, and there is no evidence suggesting that 'noticed' in (19a) is not used in this capacity. Similarly in (20b), though a theorem is not causal, we would say that Fermat's Last Theorem causes frustration if attempts to prove it are frustrating. There is no evidence to rule out that 'caused' is used in this loose way in (19b).

Moltmann's criterion for concreteness includes spatio-temporal location, and yet she admits that spatio-temporal predicates do not apply to the referents of number-of expressions (2013a: 505; 2013b: 56-7). Consider:

(21a) * The number of cats is in the bedroom.

(21b) * The cats are in the bedroom, in number.

(21c) * The cats are no longer, in number.

(21d) * The number of cats is no longer.

If number tropes were the concrete quantitative aspects of collections, they would go wherever and whenever the collections go. Even if the number of cats and the collection of cats share properties, there is no evidence that they share concrete properties. There is no evidence that number tropes are the referents of number-of expressions.

What can we conclude about number-of expressions and magnitude-of expressions? We have seen that, in come subject-predicate constructions, they do not stand for numbers, while in others, they do. Recall:

(10b) The number of planets in the solar system is surprising.

(16b) The number of planets in the solar system is a real number.

In (11b) it is that there are eight planets that is surprising, while in (16b), it is a number that is said to be real. We have seen that similar changes of reference are exhibited by other terms. Recall:

(13a) What John wants is unusual.

(13b) What John wants is small and shiny.

In (13a) it is that John wants a diamond that is unusual, while in (13b), it is the diamond that is small and shiny. This is corroborated by the fact that the following intuitively express the same propositions as (11b) and (13a), respectively:

(22a) That John wants a diamond is unusual.

(22b) That there are eight planets in the solar system is surprising.

It is plausible that the referent of 'What John wants' and 'The number of planets in the solar system' in these contexts is whatever the referent of the corresponding that-clause is. That-clauses are typically thought to denote propositions, but the following suggests that in these contexts they refer to facts:

(23a) The fact that there are eight planets in the solar system is surprising.

(23b) The fact that John wants a diamond is unusual.

(23c) * The proposition that there are eight planets in the solar system is surprising.

(23d) * The proposition that John wants a diamond is unusual.

The evidence presented in this section suggests that, in applied contexts, number-of expressions stand for facts, and the same goes for magnitude-of expressions. However, in mathematical contexts, it appears they stand for numbers. The question now is whether or not sentences such as (1a-b) provide a mathematical or non-mathematical context. If the Fregean analysis is correct, then they provide a mathematical context, since they concern numbers. However, in the previous section, we saw that there is good reason for thinking that the post-copular expressions in (1a-b) are not referring expressions, so the Fregean analysis is mistaken. But that does not yet settle the issue at hand. I leave this issue aside for the time being and return to it in §5. For now, I turn to the evaluation of some recent syntactic analyses of sentences such as (1a-b) that promise to solve the syntactic dimension of FOP.

4. Specificational Sentences

Hofweber (2005: 210-1) suggests that the unusual syntactic position of numerals in sentences such as (1a-b) is due to a widespread phenomenon called *focus*, whereby expressions are moved to unusual syntactic positions in order to emphasise certain aspects of the information expressed. Compare:

(24a) John swims quickly to shore.

(24b) The way John swims to shore is quickly.

Intuitively, (24a-b) have the same truth-conditions. However, (24a) presents the information neutrally, while (24b) emphasises the way in which John swims. To the question 'How does John swim to shore?', (24a-b) are both appropriate answers. To the question 'Where does John swim to?', (24a) is appropriate, while (24b) is not. This effect is focus.

Hofweber claims that sentences such as (1a-b) exhibit numerals in an unusual position to emphasise how many of the relevant objects there are. Indeed, to the question 'What revolves around the sun?', (1a) would be a strange answer, while 'There are eight planets in the solar system' seems perfectly appropriate. To 'How many planets are in the solar system?', both would be acceptable. Hofweber claims that, just like (24a-b), (1a) and (3a) have the same truth conditions.

However, this is not yet a solution to FOP. If (1a) and (3a) have the same truth-conditions, then 'The number of planets in the solar system' in (1a) cannot function as a referring expression. We saw in the previous section that it typically does. The unusual semantic behaviour of 'The number of planets in the solar system' in (1a) requires explanation on Hofweber's account. He doesn't offer one, so his solution is incomplete.

Felka (2014: 263-5) points out that focus is characteristic of specificational sentences, so Hofweber's examples should be taken as evidence that sentences such as (1a-b) are specificational sentences, in which case, the solution to the syntactic dimension of FOP requires an analysis of specificational sentences. Compare the following examples of predicational and specificational sentences (Mikkelsen 2011: 1806):

Predicational:

(25a) The hat I bought for Harvey is big.

(25b) What I bought for Harvey is big.

Specificational:

(26a) The director of Anatomy of a Murder is Otto Preminger.

(26b) Who I met was Otto Preminger.

Examples of predication and specification share syntactic features. (25b) and (26b) begin with an interrogative pronoun as part of what is known as a *wh-clause*. Both are *pseudoclefts*. There are predicational and specificational pseudoclefts. (25a) and (26a) begin with a headed relative clause and are plain predication and plain specificational sentences, respectively. Linguists agree that the distinction between predication and specificational is the more semantically important one. It is therefore desirable to have a semantics of predication sentences that unifies all their syntactic forms, including plain and pseudocleft, and a semantics of specificational sentences that unifies all theirs (Mikkelsen 2011: 1807; Felka 2014: 268).

Specificational sentences are used to specify who or what something is: (26a) specifies who the director of a certain movie is and (26b) specifies who the speaker met.

Specificational sentences also exhibit focus, giving emphasis to the post-copula expression. (1a-b) share these features so it is highly plausible that such sentences are specificational sentences. It is hard not to read (1a) as specifying how many planets there are, and (1b) as specifying how many kilograms the rock's mass is, and we have seen that these sentences exhibit focus.

Some theorists (Moltmann 2016; Felka 2014; Schlenker 2003) have analysed specificational sentences as question-answer pairs: the pre-copula expression as an interrogative; the post-copula an elided answer clause:

(27a) What James likes is lying down.

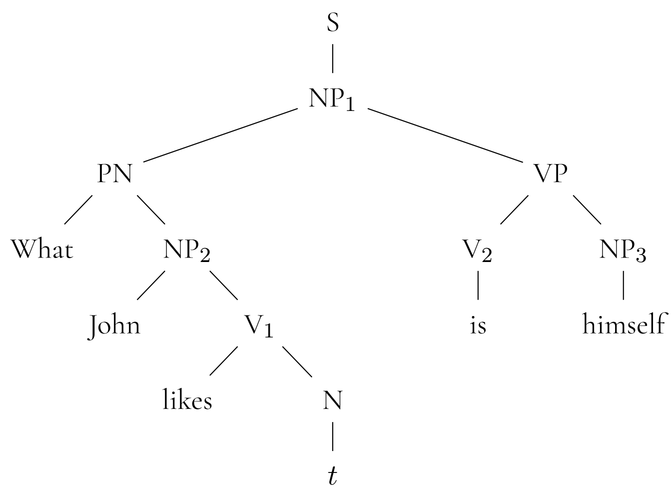
(27b) [What James likes is?] is [James likes lying down].

The evidence in support of this analysis is that it solves a syntactic puzzle. Compare:

(28a) John likes himself.

(28b) What John likes is himself.

In both, 'himself' borrows its referent from 'John'. According to binding theory, this is only possible if the two expressions stand in a specific syntactic relationship: *c-commanding*. This is cashed out in terms of dominance, represented in syntax trees. A node *x* dominates a node *y* iff *x* is above *y* and one can trace a line from *x* to *y* while only moving downwards. An expression *a* c-commands for an expression *b* iff the first branching node dominating *a* also dominates *b*, and neither *a* nor *b* dominate each other. See the representation of (28b) below:



By the above structure, attributed to (28b) by binding theory, 'John' does not c-command 'himself'. Yet, according to all the usual tests, (28b) is a case of c-commanding. This is the puzzle of connectivity.

The question-answer analysis promises to solve this because it attributes a structure to specificational sentences that meets the criteria imposed by binding theory: 'himself' in (28b) is elliptical for the answer clause 'John likes himself', whose structure is identical to (28a). The pronoun therefore borrows from the hidden occurrence of 'John', which is in the right position to lend its reference.

There is more evidence in support of the question-answer analysis, the most impressive of which is the following. Compare:

(29a) What I did then was call the grocer.

(29b) What I did then was I called the grocer. (Ross 1972, (39a), (39b))

Here the ellipsis in the post-copula clause is optional. Sometimes, it is not, but Philippe Schlenker (2003: 14) proposes a plausible explanation for this. Consider:

(30c) Who I called was the grocer.

(30d) ? Who I called then was I called the grocer.

Schlenker suggests that the repetition of the verb makes ellipsis obligatory. In contrast, there is no danger of repetition of the verb in (29b). There are good reasons for adopting the question-answer analysis, but it is important to note that the most compelling evidence suggests only that the post-copular expression is an elided answer.

Moltmann (2013a: 521-5; 2016) is a proponent of the question-answer analysis, and proposes to analyse arithmetical specificational sentences as follows:

(31a) The number of ship is seven.

(31b) [How many ships?] is [There are seven ships].

The unpacked form of 'seven' has the numeral occurring in its typical syntactic position. This fits with the semantic analysis of numerals presented in the previous section, and offers a deeper explanation of their unusual syntactic distribution: they occur in elided answer clauses in which they hold their standard position. However, Moltmann's analysis runs into a problem similar to that encountered by Hofweber. According to Moltmann, number-of expressions occur as referring expressions in most contexts, but disappear under analysis here. This phenomenon requires explanation. Moreover, no argument is offered for why the interrogative is 'How many ships?' rather than 'What is the number of ships?' The latter better resembles the original sentence, and I can think of no argument for preferring Moltmann's analysis.

It is also somewhat far fetched that apparent definite descriptions can be analysed as interrogatives at all: 'The number of ships' just doesn't seem like a question. Sensitive to this, Felka posits yet more ellipsis:

(32) [What the number of moons of Jupiter is] is [Jupiter has four moons]. (Felka 2014: 276)

Felka claims that there are other examples of embedded interrogatives where this ellipsis is optional:

(33) John revealed who the winner of the competition is.

This is an improvement on Moltmann's analysis, but it suffers from significant drawbacks. For one, it implies that definite descriptions and so-called interrogatives are semantically identical in certain contexts. Elsewhere, I present evidence to show that this implication is false, and that the syntactic distributions of these expressions differ in ways that the question-answer analysis cannot adequately explain (Knowles 2015a: 2766-2771). In this paper, however, I want to undermine the question-answer analysis on different grounds. In the following section, I present a semantics of interrogatives that suggests we should instead think of expressions like 'Who the winner of the competition is' as answer clauses rather than question clauses. Along with the evidence in favour of accepting that the post-copular expressions of specificational sentences are elided answers, this suggests an analysis of specificational sentences as answer-answer pairs, leading naturally to the semantics of specificational presented in §6.

5. The Semantics of Interrogatives

In their seminal paper, Jeroen Groendjik and Martin Stokhof (1997) motivate a simple and explanatorily powerful semantics of interrogatives that emerges from three plausible principles:

(i) An answer to a question is a proposition.

(ii) The possible answers to a question are an exhaustive set of mutually exclusive possibilities.

(iii) To know the meaning of a question is to know what counts as an answer to it.

The first principle falls out of two plausible considerations: first, that the proper linguistic vehicle for an answer is a declarative sentence; second, that a question asks for a certain piece of information. The second principle puts a constraint on propositions that count as answers to a question: the truth of one entails that the rest are false, so they logically exclude one another, and they are exhaustive, in that the set of answers to a question completely fills the logical space defined by the question. The picture that emerges is that the semantic value of an interrogative is a partition of logical space. The set of possible worlds at which the semantic presuppositions of the question are true is the logical space defined by the question and the cells of the partition are the subsets, each constituting a proposition that is an exhaustive possible answer to the question.

The final principle parallels one of the foundational principles of truth-conditional semantics of indicatives. Just as we assume that to know the meaning of an indicative is to know the conditions under which it is true, we assume that to know the meaning of an interrogative is to know the conditions under which it is answered. On this view, the intension of an interrogative is a partition, and its extension relative to a world is the proposition that is its unique and true answer at that world. Call this the *partition analysis*.

As well as its intuitive appeal, the partition analysis has considerable explanatory power. It allows us to account for apparent meaning relations between interrogatives.

For example, the interrogatives 'Who came to the party?' and 'Which women came to the party?' stand in a sort of entailment relation in that a complete answer to the former entails a complete answer the latter. On the partition analysis, this can be dealt with in familiar terms: question P? entails question Q? iff each cell of the partition denoted by P? is included in some cell of the partition denoted by Q?. The analysis also makes sense of cases in which an indirect interrogative is embedded in an intensional verb, such as 'wonder':

(34a) Cooper wondered who killed Laura Palmer.

(34b) * Cooper wondered that Bob killed Laura Palmer.

Because 'wonder' is an intensional verb, its semantic value takes the intension of the indirect interrogative as an argument. A partition is an unresolved entity, so to speak, and truth and falsity cannot be predicated of it, so it seems the appropriate object for the relation expressed by the verb. That 'wonder' selects for an object such as this can be seen from (34b): it cannot take a fact or proposition-denoting expression as argument because facts are 'resolved' and propositions are bear truth values.

However, the partition analysis has trouble accounting for interrogatives embedded in non-intensional contexts. Consider the following invalid argument (some of which is adapted from Ginzburg 1995):

(35a) Cooper discovered an interesting question.

(35b) The question he discovered was who killed Laura Palmer.

(35c) Therefore, Cooper discovered who killed Laura Palmer.

The partition-based explanation for the invalidity of this argument would presumably have to rest on the claim that 'Who killed Laura Palmer' in (35b) contributes its intension, a partition, while in (35c) it contributes its extension, namely whichever proposition constitutes its true and exhaustive answer at the relevant world. The problem arises when we test these claims directly. It turns out that indirect interrogatives cannot occupy positions that allow for proposition-denoting expressions:

(36a) Cooper believed an interesting proposition.

(36b) ? The proposition Cooper believed was who killed Laura Palmer.

(36c) ? Therefore, Cooper believed who killed Laura Palmer.

They do, however, occupy positions reserved for fact-denoting expressions:

(37a) Cooper discovered an interesting fact.

(37b) The fact Cooper discovered was who killed Laura Palmer.

(37c) Therefore, Cooper discovered who killed Laura Palmer.

Moreover, we have already seen in §3 that they serve as subject in subject-predicate constructions in which it is most plausible to assume that they denote a fact. For example:

(38) What John wants reflects badly on his character.

It is the fact that John wants ice cream that reflects badly on his character, not the proposition or the ice cream itself. We see the same patterns in nominalizations of indicative sentences, or that-clauses. Consider:

(39a) Cooper discovered an interesting fact.

(39b) The fact was that Bob killed Laura Palmer.

(39c) Therefore, Cooper discovered that Bob killed Laura Palmer.

Again, that-clauses can occur in subject-predicate constructions in which the most natural reading is that they stand for facts:

(40) That John likes ice cream reflects badly on his character.

Again, it is the fact that John likes ice cream that reflects badly on his character.

All of this suggests that we need an account of indirect interrogatives according to which, like that-clauses, they can refer to facts. Elsewhere, I have suggested one way of achieving this (Knowles 2015a: 2771-2773). If we understand facts as the parts of worlds that true propositions correspond to, then we can assign to each world the set of facts that form part of that world (the obtaining facts), and the set of facts that do not (the non-obtaining facts). Then we can assume that a sentence extensionally denotes the fact that makes it either true or false at that world, rather than its truth-value at that world, and treat indirect interrogatives as syntactically derived nominalizations of those sentences that refer to those facts. There is not space to present the proposal in detail here. Instead, I want to demonstrate that it yields an account of interrogatives that is both consistent with the motivations for principles (i)-(iii) above, and with the data concerning embedding in factive contexts.

In short, the view is as follows. Facts are assigned as the extension (and referent) of indirect interrogatives. From this, it follows that the intension of an indirect interrogative is a function from worlds to facts. Recall that the motivation for principle (i) was that questions are a request for information, and that the proper vehicle for an answer is an indicative sentence. Propositions are valuable for answering questions, but arguably not in and of themselves. Plausibly, they are well-suited to this purpose only insofar as they can tell us about what facts obtain in the actual world. Moreover, they are dispensable to this purpose. If I ask 'Where are my keys?', and then reach into my pocket and find them, I might say 'That answers my question'. A more direct indication of the facts than via the communication of a true proposition can clearly provide answers to questions. It seems plausible, then, to take facts to be the primary answers to questions, and propositions merely as a convenient vehicle for representing them. As for the claim that sentences are the proper vehicles for answers, if facts are the extensions of sentences, then the most appropriate way of indicating the relevant fact will often be to utter the appropriate indicative. So, let us change (i) to (iii) accordingly:

(i) An answer to a question is a fact.

(ii) The possible answers to a question are an exhaustive set of mutually exclusive possibilities.

(iii) To know the meaning of a question is to know what counts as an answer to it.

This allows us to account for the occurrences of interrogatives embedded in factive verbs, such as 'know'. On the present account, fact-denoting that-clauses and indirect interrogatives both denote facts. The validity of (37a-c) and (39a-c) is explained by the fact that these expressions share a denotation:

(37a) Cooper discovered an interesting fact.

(37b) The fact he discovered was who Killed Laura Palmer.

(37c) Therefore, Cooper discovered who killed Laura Palmer.

(39a) Cooper discovered an interesting fact.

(39b) The fact was that Bob killed Laura Palmer.

(39c) Therefore, Cooper discovered that Bob killed Laura Palmer.

When indirect interrogatives are embedded in intensional contexts such as 'wonder', it is their intension that is contributed. Recall that the object is required to be such that it is unresolved and cannot be true or false. A function from worlds to facts is unresolved in that it doesn't settle the question of which fact actually obtains, and truth and falsity cannot be attributed to it, so the acceptability of (34a) is explained:

(34a) Cooper wondered who killed Laura Palmer.

Finally, propositions are not assigned as either the intensions or extensions of indirect interrogatives, so we expect that they would not be able to occupy positions reserved for proposition-denoting expressions. Recall:

(36a) Cooper believed an interesting proposition.

(36b) ? The proposition he believed was who killed Laura Palmer.

(36c) ? Cooper believed who killed Laura Palmer.

There is good reason to think that indirect indicatives refer to facts. There is one remaining issue that needs addressing, however. Any plausible analysis of interrogatives must explain the relationship between indirect interrogatives, such as 'Where my keys are', and direct interrogatives, such as 'Where are my keys?' (Karttunen 1977: 3-4). On the partition analysis, partitions are assigned as the semantic values of both direct and indirect interrogatives, and I want to maintain this assignment with respect to the former. That a question divides logical space into propositions that represent the ways the world would have to be for the question to be answered is intuitive. In light of this, however, I must relinquish the assumption that expressions such as 'Where my keys are' are interrogatives at all. Instead, I propose we understand them as answer clauses. By an 'answer clause', I mean an expression that refers to the fact that answers the corresponding question. There are two good independent reasons for adopting this view. First, indirect interrogatives sound like answers. One can imagine a particularly facetious person providing (41b) as an answer to (41a):

(41a) What does John want?

(41b) What John wants.

It is not a useful way of communicating the answer, but it does point to the right fact. The second reason is that the account provides an elegant explanation of the relationship between indirect and direct interrogatives while still accounting for their difference in structure. Indirect interrogatives denote the facts that answer the questions posed by direct interrogatives; they differ in structure because they belong to different semantic types. I am now in a position to present my semantic account of specificational sentences.

6. Solutions and Conclusions

My account of indirect interrogatives as answer clauses that refer to facts leads naturally to a plausible analysis of specificational sentences as pairs of answer clauses. We have seen that there is good reason to think that the pre-copular expressions of specificational sentences refer to facts. If we add to that the claim that the elided indicative answer clause in post-copular position also refers to the fact that makes the proposition it expresses true, then we get an analysis according to which specificational sentences identify facts. Call this the 'fact analysis' (FA).

I can now state the truth-conditions of a specificational sentence. (42b) indicates the syntactic analysis, while (42c) specifies the truth-conditions:

(42a) The one who knocks is Walter White.

(42b) [The one who knocks] is [Walter White knocks]

(42c) 'The one who knocks is Walter White' is true iff: the fact that uniquely and exhaustively answers the question 'Who is the one who knocks?' is identical with the fact that Walter White knocks.

Perhaps more needs to be said about what I mean by facts. I do not have the space or the inclination to provide a metaphysical account of facts here. In this context, they are to be understood primarily as the referents of fact-referring expressions. I suggested in the previous section that we think of them as the parts of worlds in that true propositions correspond to. So, a proposition true at a world corresponds to a fact that it part of that world, and a proposition that is false at that world corresponds to a fact that is not part of that world, but part of some other world. If (42a) is true, the fact that Walter White knocks is the part of the world in virtue of which the proposition that Walter White knocks is true. Moreover, if (42a) is true, the fact that Walter White knocks is also the part of the world that uniquely and exhaustively settles the question 'Who is the one who knocks?'. This happens just in case Walter White is the only one who knocks. Intuitively, then, FA gets the truth-conditions for (42a) right.

Doesn't the answer-answer analysis run counter to the evidence presented in favour of the question-answer analysis presented in §4? No. The evidence suggests only that the post-copular clause is an elided answer, and FA is consistent with this. Schlenker (2003) does provide evidence that he takes to suggest that the pre-copular expressions of specificational sentences are interrogatives, but it all presupposes that expressions like 'Where my keys are' are interrogative clauses when they occur in other, non-specificational contexts. In the previous section, I provided good reasons for thinking that such expressions are fact-referring answer clauses across the board. For this reason, I do not find Schlenker's evidence compelling.

Though they are different, FA is similar to the analysis of specificational sentences defended by Schlenker (2003), who also takes them to be identity statements. He is a proponent of the question-answer analysis and the partition view of interrogatives. He claims that specificational sentences are identity statements that equate the proposition that, according to him, amounts to the exhaustive unique true answer to the relevant question. We have seen the problems that the partition view faces. In particular, it has problems explaining the occurrence of indirect interrogatives in factive contexts, and their failure to occur in positions reserved for proposition-denoting expressions. These problems extend to cases involving specificational sentences. On the one hand, Schlenker has difficulty explaining the validity of arguments such as the following:

(43a) Mary discovered an interesting fact, namely what John had for pudding last night.

(43b) What John had for pudding last night is ice cream.

(43c) Therefore, Mary discovered that John had ice cream for pudding last night.

The validity of this argument seems to require that facts are referred to throughout. On Schlenker's account, however, it is difficult to makes sense of (43a), and more generally to make sense of the validity of the above argument. On the other hand, Schlenker's account also has problems accounting for the infelicity of the following:

(44a) * Mary believed an interesting proposition, namely what John had for pudding last night.

(44b) What John had for pudding last night is ice cream.

(44c) Mary believed that John had ice cream for pudding last night.

On Schlenker's view, the above should be acceptable, but it isn't. This is to be expected on FA, since propositions are not assigned to either the pre- or post-copular expressions of specificational sentences. (See Knowles 2015a: 2772-2773 for more reasons to prefer FA to Schlenker's view).

I will now state the truth-conditions of (1a-b) and discuss what we can conclude from them with regards the arguments mentioned in the introduction.

(1a''') 'The number of planets in the solar system is eight' is true iff: the fact that uniquely and exhaustively answers the question 'How many planets are there in the solar system?' / 'What is the number of planets in the solar system?' is identical to the fact that there are eight planets in the solar system.

(1b''') 'The mass of Jupiter in kilograms is 1.8986×10^{27}' is true iff: the fact that uniquely and exhaustively answers the question 'What is the mass of Jupiter in kilograms?' is identical to the fact that the mass of Jupiter in kilograms is 1.8986×10^{27}.

The first thing to note here is that the truth-conditions of (1a-b) do not mention mathematical objects. They mention certain kinds of facts. In the case of sentences such as (1a), what we might call 'cardinality facts'; in cases such as (1b), what we might call 'magnitude facts'. So, the Fregean analysis does not provide an adequate account of the semantics of these sentencnes, and the truth-conditions of sentences such as (1a-b) do not appear to involve mathematical objects. On the face of it, then, proponents of easy arguments and indispensability arguments cannot appeal to applied arithmetical language.

Is there a way for the realist to salvage these arguments? There are two routes to take. The first is to focus on different kinds of mathematical sentences. This will be less successful in the case of easy arguments. Easy arguments are effective because they start from a highly intuitive premise, that sentences such as (1a-b) are true. If the realist were to instead appeal to pure mathematical language in support of an easy argument, I suspect that intuitions would not be as strongly in their favour. For the proponent of the indispensability argument, however, appealing to the applications of pure mathematical language in science is more promising. For instance, pure arithmetical language is intuitively about numbers, and enjoys many applications in science. However, if this paper has shown anything, it is that semantic assumptions such as this should not be taken lightly. A successful indispensability argument would need, *inter alia*, to provide evidence in favour of its semantic assumptions.

The other route is instead to maintain on metaphysical grounds that the truth of applied arithmetical sentences, such as (1a-b), does require mathematical objects to exist. Granted, the truth-conditions of (1a-b) only mention cardinality facts and magnitude facts; but they do not say anything about the nature of these facts. The realist could argue that, in order for such facts to obtain, there must be mathematical objects. The view that magnitude facts consist in physical objects bearing relations to numbers has a name, 'Heavy Duty Platonism', and might naturally be thought to go hand in hand with the corresponding view about cardinality facts. Heavy Duty Platonism is not a popular view, but, recently, I (Knowles 2015b) have shown that all the arguments against it presented and alluded to in the literature fail. Again, adding a defence of Heavy Duty Platonism to an easy argument will render it no longer worthy of its name, but defending Heavy Duty Platonism is a promising means for the realist to defend an indispensability argument that appeals to applied arithmetical language.

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Cited works

Knowles, R. 2015a. What 'the number of planets is eight' means. *Philosophical Studies* 172: 2757-2775.

Knowles, R. 2015b. Heavy Duty Platonism. *Erkenntnis* 80: 1255-1270.

Barwise, F. and Cooper, R. 1981. Generalised Quantifiers and Natural Language. *Linguistics and Philosophy* 4: 159-219.

Dolby, D. 2009. The Reference Principle: A Defence. *Analysis* 69: 296-96.

Felka, K. 2014. Number Words and Reference to Numbers. *Philosophical Studies* 168: 261-82.

Field, H. 1980. *Science Without Numbers: A Defence of Nominalism*. Princeton University Press.

Field, H. 1989. *Realism, Mathematics, and Modality*. Oxford, UK: Blackwell Publishing.

Frege, G. 1884. *The Foundations of Arithmetic: A Logico-Mathematical Enquiry into the Concept of Number*. Translated by Austin, J. 1953. Oxford: Blackwell Publishing.

Ginzburg, J. 1995. Resolving Questions II. *Linguistics and Philosophy* 18: 567-609.

Groenendijk, J. and Stokhof, M. 1997. Questions. In van Benthem, J. and ter Meulen, A. (eds.) *Handbook of Logic and Language*. MIT Press: 1055-1124.

Hintikka, J. 1974. Questions about Questions. In Munitz, M. and Unger, P. (eds.) *Semantics and Philosophy*. New York: New York University Press.

Hofweber, T. 2005. Number Determiners, Numbers, and Arithmetic. *Philosophical Review* 114:179-225.

Karttunen, L. 1977. Syntax and Semantics of Questions. *Linguistics and Philosophy* 1: 3-44.

Keenen, L. and Westerstahl, D. 1997. Generalized Quantifiers in Linguistics and Logic. In Benthem, J. and Meulen, A. (eds.) *Handbook of Logic and Language*. Cambridge: MIT Press.

Künne, W. 2006. Properties in Abundance. In Strawson, P. F. (ed.) *Universals, Concepts, and Qualities: New Essays on the Meaning of Predicates and Abstract Entities*. Aldershot: Ashgate: 249-300.

Lahiri, U. 2002. *Questions and Answers in Embedded Contexts*. Oxford: Oxford University Press.

Mikkelsen, L. 2011. Copular Clauses. In Maienborn, C. Portner, P, and von Heusinger, K. (eds.) *Semantics: An International Handbook of Natural Language Meaning* (Vol. 2). Berlin: Mouton de Gruyter: 1805-29.

Moltmann, F. 2016. The number of Planets, a Number-Referring Term? In Ebert, P. and Rossberg, M. (eds.) *Abstractionism: Essays in Philosophy of Mathematics*. Oxford University Press: 113-129.

Moltmann, F. 2013a. Reference to Numbers in Natural Language. *Philosophical Studies* 162: 499-536.

Moltmann, F. 2013b. *Abstract Objects and the Semantics of Natural Language*. Oxford: Oxford University Press.

Montague, R. 1974. The Proper Treatment of Quantification in Ordinary English. In Thomason, R. (ed.) *Formal Philosophy*. New Haven: Yale University Press.

Mostowski, A. 1957. On a Generalization of Quantifiers. *Fundamenta Mathematicae* 44: 12-36.

Oliver, A. 2005. The Reference Principle. *Analysis* 65: 177-87.

Prior, A. N. 1971. *Objects of Thought*. Oxford: Clarendon Press.

Quine, W. V. 1948. On What There Is. *The Review of Metaphysics* 2: 21-38.

Rosefeldt, T. 2008. 'That'-Clauses and Non-Nominal Quantification. *Philosophical Studies* 137: 301-33.

Ross, J. 1972. Act. In Davidson, D. and Harman, G. (eds.) *Semantics of Natural Language*. Dordrecht: D. Reidel and Company.

Schlenker, P. 2003. Clausal Equations: A Note on the Connectivity Problem. *Natural Language and Linguistic Theory* 21: 157-214.

Sellars, W. 1960. Grammar and Existence: a Preface to Ontology. *Mind* 69: 499-533.